J. Fluid Mech. (1968), *vol.* 32, *part* 4, *pp.* 801–808
Printed in Great Britain

The development of three-dimensional wave-packets in unbounded parallel flows

By M. GASTER AND A. DAVEY
National Physical Laboratory, Teddington, Middlesex

(Received 5 December 1967)

In this paper we examine the stability of a two-dimensional wake profile of the form $u(y) = U_\infty(1 - r\, e^{-sy^2})$ with respect to a pulsed disturbance at a point in the fluid. The disturbed flow forms an expanding wave packet which is convected downstream. Far downstream, where asymptotic expansions are valid, the motion at any point in the wave packet is described by a particular three-dimensional wave having complex wave-numbers. In the special case of very unstable flows, where viscosity does not have a significant influence, it is possible to evaluate the three-dimensional eigenvalues in terms of two-dimensional ones using the inviscid form of Squire's transformation. In this way each point in the physical plane can be linked to a particular two-dimensional wave growing in both space and time by simple algebraic expressions which are independent of the mean flow velocity profile. Computed eigenvalues for the wake profile are used in these relations to find the behaviour of the wave packet in the physical plane.

1. Introduction

Unbounded flows, such as jets and wakes, are very unstable at quite moderate Reynolds numbers and are easily disturbed from the steady laminar state. Small input disturbances generate motions composed of linear combinations of the simple waves of linear stability theory. One of the simplest forms of excitation is a pulsed source at some point in the flow. The wave packet, which arises from this disturbance through selective amplification and interference of the various modes generated by the source, is of some practical interest in transition studies. Natural transition often occurs through the formation and growth of turbulent spots which are presumably initiated by these linear wave packets.

The motion generated by a pulse perturbation (delta function) can be found by evaluating a double integral of all possible eigenmodes over all wave-numbers. In general it is not possible to evaluate this integral in a closed form, but since most interest is centred on the form of the solution far from the source, asymptotic expansion techniques can be employed. Previous solutions by Brooke Benjamin (1961) and by Criminale & Kovasznay (1962) used expansions about a fixed point in the wave-number frequency domain. In this paper we use more general asymptotic expansions which enable us to link the motion at any point in the physical plane to a particular eigenmode having complex values of wave-number and frequency. It would in principle be possible to compute such modes and thus examine the structure of the disturbed flow in the physical plane, but such computations would be very time consuming and it is thus worth considering possible simplifying approximations.

In a previous paper discussing the wave packet in a boundary layer (Gaster 1968), it was shown that eigenvalues for any mode with complex wave-numbers could be obtained from purely temporally increasing modes provided the imaginary part of the wave-numbers were small. This small amplification approximation coupled with Squire's (1933) transformation, which relates any temporally increasing oblique wave to a two-dimensional one at a lower Reynolds number, enabled the motion over the whole region of the wave packet to be obtained in terms of the two-dimensional temporally growing waves usually discussed in stability theory.

Very unstable flows admit waves with large amplification rates and the expansion techniques employed for the boundary-layer problem cannot be used. However, since these very unstable waves are almost unaffected by viscosity at high Reynolds numbers, the eigenvalues are adequately defined by the inviscid disturbance equation. The inviscid form of Squire's transformation allows *all* three-dimensional waves, including those with complex wave-numbers, to be related to two-dimensional ones. This simplification, which is used here, enables the asymptotic expansions to be evaluated quite generally. The motion at any point in the physical plane can then be found in terms of parameters defining two-dimensional modes which have complex values of both wave-number and frequency. This analysis is applied to the wake profile with the aid of a digital computer.

2. Formulation of the problem

We suppose that the mean flow is approximately parallel and we linearize the equations of motion with respect to the disturbance to obtain a set of homogeneous equations which admit solutions of the form

$$\left.\begin{aligned} \hat{u}(y;x,z,t) &= u(y;a,b,\omega)\exp\{i(ax+bz-\omega t)\},\\ \hat{v}(y;x,z,t) &= v(y;a,b,\omega)\exp\{i(ax+bz-\omega t)\},\\ \hat{w}(y;x,z,t) &= w(y;a,b,\omega)\exp\{i(ax+bz-\omega t)\}\\ \text{and}\qquad \hat{p}(y;x,z,t) &= p(y;a,b,\omega)\exp\{i(ax+bz-\omega t)\},\end{aligned}\right\} \tag{1}$$

where $\hat{u}, \hat{v}$ and $\hat{w}$ are the perturbation velocities in directions x, y and z respectively; and $\hat{p}$ is the pressure perturbation. The wave-numbers a and b, in the x- and z-directions, may be complex.

The disturbance produced by some sort of pulse excitation at a point on the y-axis will be given in the form of a double integral to be evaluated over a and b.

Following Gaster (1968) we obtain

$$\hat{v}(y;x,z,t) \sim \int_{C_a}\int_{C_b} \exp\{i(ax+bz-\omega t)\}\,da\,db, \tag{2}$$

where the eignevalue ω is a function of both a and b.

Taking the first term of the asymptotic expansion for large t, x/t and z/t being treated as of order unity, and neglecting a phase factor we obtain the form

$$v \sim \frac{\exp\{i[a^*x/t + b^*z/t - \omega(a^*, b^*)]t\}}{t\left[\dfrac{\partial^2\omega(a^*, b^*)}{\partial a^2}\dfrac{\partial^2\omega(a^*, b^*)}{\partial b^2} - \left\{\dfrac{\partial^2\omega(a^*, b^*)}{\partial a\,\partial b}\right\}^2\right]^{\frac{1}{2}}}, \tag{3}$$

where a^* and b^* are chosen such that

$$\frac{x}{t} = \frac{\partial\omega}{\partial a}(a^*, b^*) \quad \text{and} \quad \frac{z}{t} = \frac{\partial\omega}{\partial b}(a^*, b^*), \tag{4}$$

and a^*, b^* and $\omega(a^*, b^*)$ are complex.

Note that the above result is only valid if the denominator is not small.

3. Squire's transformation

Using the perturbations given in (1) the linearized equations of motion for a two-dimensional parallel mean flow become

$$\frac{-i}{aR}\{u'' - (a^2+b^2)u\} = -\frac{iv}{a}\frac{dU}{dy} + p + \left(U - \frac{\omega}{a}\right)u,$$

$$\frac{-i}{aR}\{v'' - (a^2+b^2)v\} = -\frac{ip'}{a} + \left(U - \frac{\omega}{a}\right)v,$$

$$\frac{-i}{aR}\{w'' - (a^2+b^2)w\} = \frac{b}{a}p + \left(U - \frac{\omega}{a}\right)w,$$

together with the equation of continuity

$$i(au + bw) + v' = 0.$$

R is some convenient Reynolds number of the mean flow; the non-dimensional wave-numbers a and b, and the frequency ω, being normalized on the basis of the chosen scale of length and velocity. Primes denote differentiation with respect to y and the mean flow is $U(y)$. These equations can be reduced to the form

$$\left(U - \frac{\omega}{a}\right)\{v'' - (a^2+b^2)v\} - U''v = \frac{-i}{aR}\{v^{\text{iv}} - 2(a^2+b^2)v'' + (a^2+b^2)^2 v\}. \tag{5}$$

The Orr–Sommerfeld equation which describes two-dimensional waves of the form $\hat{v}(y; x, t) = v(y; \alpha, \beta)\exp\{i(\alpha x - \beta t)\}$ is

$$(U - \beta/\alpha)(v'' - \alpha^2 v) - U''v = \frac{-i}{\alpha R_2}(v^{\text{iv}} - 2\alpha^2 v'' + \alpha^4 v), \tag{6}$$

where R_2 is the Reynolds number of the two-dimensional flow. Squire (1933) showed that equations (5) and (6) are equivalent when

$$\alpha^2 = a^2 + b^2, \quad \beta/\alpha = \omega/a \quad \text{and} \quad \alpha R_2 = aR. \tag{7}$$

Thus for temporally growing waves, which have a and α real, we can relate any three-dimensional mode to a two-dimensional one at a lower Reynolds number. However, in the more general case when wave-numbers are complex, this simple transformation shows that any three-dimensional mode can only be related to a two-dimensional mode with a complex Reynolds number aR/α.

In unbounded flows viscous effects may usually be neglected if the Reynolds number is sufficiently large, and it can be shown that solutions of the Orr–Sommerfeld equation satisfy the second-order inviscid equation in the limit of vanishing viscosity except for those eigenfunctions which are singular. These singular solutions arise at the critical layer where $\{U(y)-(\beta/\alpha)_r\}$ is zero if $(\beta/\alpha)_i$ is also zero. Thus solutions of the inviscid equation with zero $(\beta/\alpha)_i$ are not valid approximations to solutions of the full Orr–Sommerfeld equation and they cannot therefore be used in this analysis.

In the present problem where we have a complex 'Reynolds number' it can again be shown that solutions of the inviscid equation tend to solutions of the full equations at large Reynolds numbers provided a_r is not small. Some care is needed here as the inviscid equation also admits conjugate solutions. Tollmien (1929) showed that in the usual case of positive real Reynolds number the self-excited modes, which have $(\beta/\alpha)_i$ positive, must be chosen. In the case of a large complex parameter we again find that the self-excited waves must be taken when $(a/\alpha)_r$ is positive, but the damped mode is the proper branch when $(a/\alpha)_r$ is negative. When $(a/\alpha)_r$ is very small the inviscid solutions do not represent solutions to the full equations and we must avoid using these modes in this discussion.

The three-dimensional wave

$$\sim \exp\{i(ax+bz-\omega t)\}$$

at large Reynolds numbers can be linked with the inviscid two-dimensional mode

$$\sim \exp\{i(\alpha x-\beta t)\},$$

where $\alpha^2 = a^2+b^2$ and $a\omega = \alpha\beta$. In general a, b, α, β and ω are complex and the restrictions mentioned previously apply.

If the two-dimensional waves are described by the eigenmode relation

$$\beta = \alpha E(\alpha), \tag{8}$$

the related three-dimensional modes are given by

$$\omega = aE(\alpha) \quad \text{or} \quad \omega = aE(a^2+b^2)^{\frac{1}{2}}, \tag{9}$$

where the functions E in (8) and (9) are identical.

4. The solution in the physical plane

The link between wave-number space and the physical plane is provided by (4) and (9),

$$\frac{x}{t} = \frac{\partial\omega}{\partial a}(a^*, b^*) \quad \text{and} \quad \frac{z}{t} = \frac{\partial\omega}{\partial b}(a^*, b^*),$$

where $$\omega = aE(a^2+b^2)^{\frac{1}{2}}.$$

Using these eigenvalue relationships between two- and three-dimensional waves we can obtain an equation for x/t and z/t in terms of a^*, b^* and E.

Dropping the asterisk we obtain

$$\frac{x}{t} = E + \frac{a^2 E'}{(a^2+b^2)^{\frac{1}{2}}} \quad \text{and} \quad \frac{z}{t} = \frac{abE'}{(a^2+b^2)^{\frac{1}{2}}}. \tag{10}$$

Primes denote differentiation with respect to α and $E = \beta/\alpha$. Eliminating a and b from (10) provides the equation linking the physical plane to the α-plane,

$$\left(\frac{x}{t} - \frac{\beta}{\alpha}\right)\left(\frac{x}{t} - \frac{d\beta}{d\alpha}\right) + \left(\frac{z}{t}\right)^2 = 0. \tag{11}$$

By splitting this complex equation into real and imaginary parts and noting that x/t and z/t are purely real we can obtain x/t and z/t in terms of the real and imaginary parts of β/α and $d\beta/d\alpha$.

The equation defining the disturbance in the physical plane (3) may be written

$$v \sim \frac{\exp\{i[a(a^2+b^2)^{\frac{1}{2}} E']t\}}{t\left[\left(\dfrac{2a^2-b^2}{a^2+b^2}\right)E'^2 + \dfrac{a^2}{(a^2+b^2)^{\frac{1}{2}}}E'E''\right]^{\frac{1}{2}}}, \tag{12}$$

which becomes

$$\frac{\alpha \exp\left\{i\alpha\left[\left(\dfrac{x}{t} - \dfrac{\beta}{\alpha}\right)^2 + \left(\dfrac{z}{t}\right)^2\right]^{\frac{1}{2}} t\right\}}{t\left[\left(\alpha\dfrac{x}{t} - \beta\right)\dfrac{d^2\beta}{d\alpha^2} - \left(\dfrac{x}{t} - \dfrac{d\beta}{d\alpha}\right)^2 - \left(\dfrac{z}{t}\right)^2\right]^{\frac{1}{2}}}. \tag{13}$$

Equations (11) and (13) define the motion at any point in the physical $(x/t, z/t)$-plane in terms of the eigenvalues of two-dimensional modes.

5. Results

The eigenvalues and other parameters derived from eigenvalues arising in (11) and (13) were computed for the wake profile

$$u(y) = U_\infty\{1 - 0{\cdot}692 \exp(-0{\cdot}693 y^2)\}.$$

This profile was used by Sato & Kuriki (1961) for calculating temporally growing waves which they compared with their experiments. Davey also used this profile for the determination of spatially growing eigenvalues.

Figure 1 shows the α-plane with contours of β_r and β_i for the self-excited branch with $(\beta/\alpha)_i > 0$. These modes transform into growing oblique waves only when $(a/\alpha)_r$ is positive, that is to the right of the line OD which is defined by the condition

$$\mathscr{R}\left\{\frac{\dfrac{x}{t} - \dfrac{\beta}{\alpha}}{\left[\left(\dfrac{x}{t} - \dfrac{\beta}{\alpha}\right)^2 + \left(\dfrac{z}{t}\right)^2\right]^{\frac{1}{2}}}\right\} = 0.$$

Only the complex conjugate of points to the left of OD can be transformed to three-dimensional waves, but as these modes are highly damped they are of no particular interest in the present context.

The condition $(\partial\beta_i/\partial\alpha_r) = 0$, which transforms into $z/t = 0$ in the physical plane, is shown by the line AB. To the right of this line $(z/t)^2$ is negative and no meaningful solutions exist. The two points A and B transform into points of zero amplification rate and the region between them contains amplified disturbances. The neutral boundaries of the wave packet which pass from A to O and from B to D are also shown on figure 1. Thus the region enclosed by the lines A–B–D–O contains all the two-dimensional eigenvalues that transform into amplified

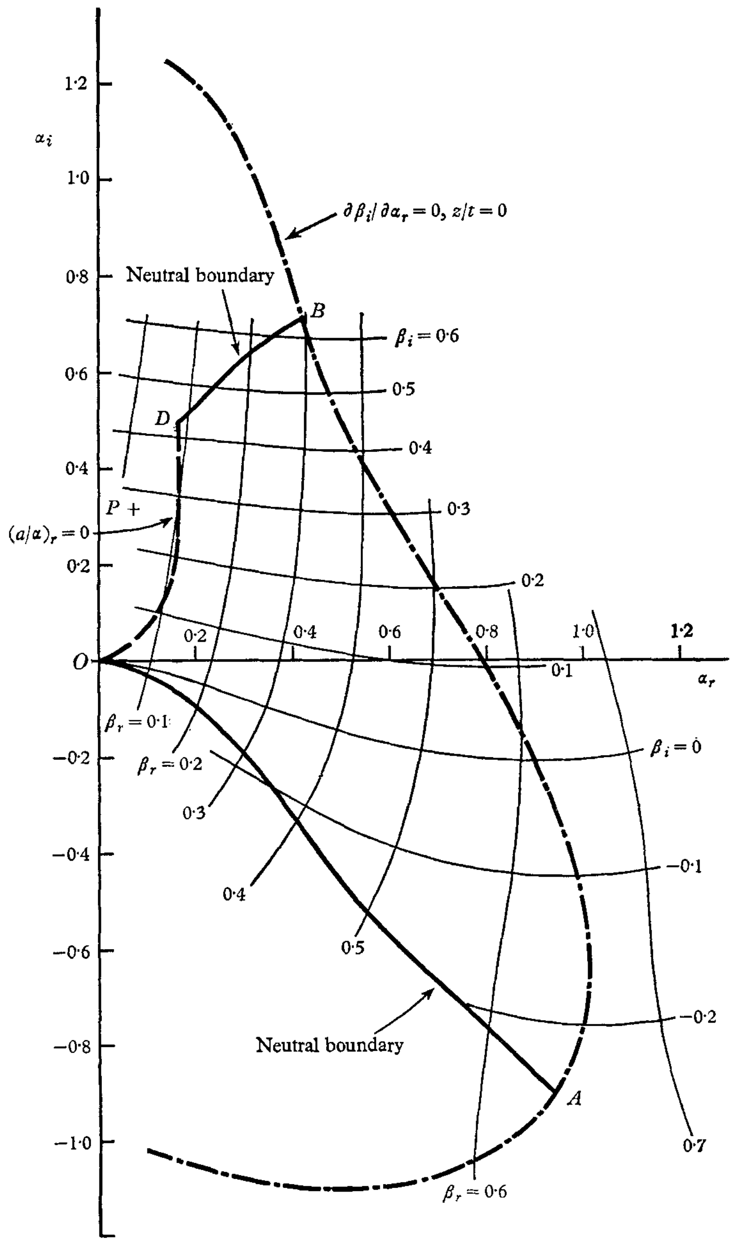

FIGURE 1. Complex α-plane.

regions in the physical plane. Figures 2 and 3 indicate how this region maps onto the physical plane, figure 2 showing lines of constant α_r and α_i and figure 3 contours of constant amplification rate.

Some care is needed in applying (11) and (13) to the inviscid two-dimensional eigenvalues, as there are regions where the analysis is invalid. For example, the inviscid eigenvalues are not good approximations to those of the full equations of motion when $(\beta/\alpha)_i$ is small and thus the solution in the region around O, in

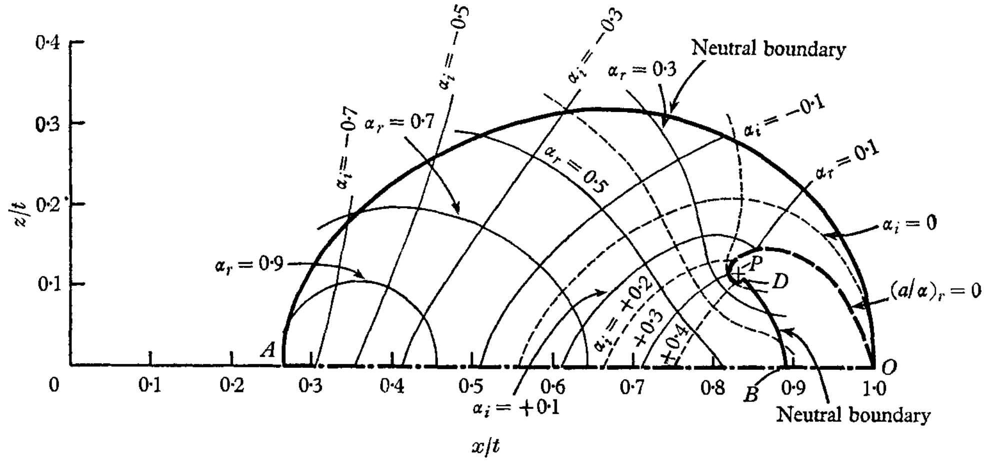

FIGURE 2. Contours of α_r and α_i on the physical plane.

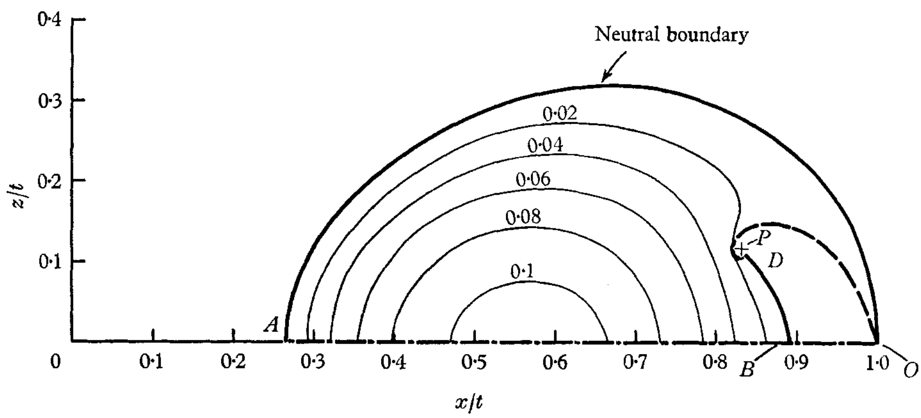

FIGURE 3. Physical plane showing amplification contours.

both the α-plane and the physical plane, should be treated with caution. Viscosity must also play a dominant role in the region of small $(a/\alpha)_r$ and the discontinuous behaviour along OD, shown in figure 3, will no doubt be smoothed out in any real physical problem. The asymptotic expansion (13) is clearly invalid where the denominator is zero, and more terms need to be used to describe the behaviour there. The denominator of (13) does vanish at a point P, which is marked on figures 1, 2 and 3, but since this relates to a damped region of the wave packet is it not of great significance in the present example.

6. Discussion

The shape and amplification pattern of a wave packet has been obtained for a typical wake profile. The amplified region is roughly elliptical except near the leading edge where the present inviscid theory predicts a discontinuity. Although viscosity would smooth out such a discontinuity it seems clear that there will be large velocity gradients present in this region. These large gradients could introduce non-linear effects and one might expect this forward region of the wave packet to be the first to break down into some form of turbulence.

We wish to thank Prof. J. T. Stuart for some helpful discussions. This work was carried out as part of the general research programme of the National Physical Laboratory.

REFERENCES

Benjamin, T. Brooke, 1961 *J. Fluid Mech.* **10**, 401.
Criminale, W. O. & Kovasznay, L. S. G. 1962 *J. Fluid Mech.* **14**, 59.
Gaster, M. 1968 *J. Fluid Mech.* **32**, 173.
Sato H. & Kuriki, K. 1961 *J. Fluid Mech.* **11**, 321.
Squire, H. B. 1933 *Proc. Roy. Soc.* A **142**, 621.
Tollmien, W. 1929 *Nachr. Ges. Wiss. Göttingen*, 21–44.